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Supersymmetric quantum mechanics defined as sesquilinear forms

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Abstract. A formulation of supersymmetric quantum mechanics in terms of sesquilinear forms is presented. Otherwise occurring rather complicated domain questions are solved in a simple and natural way. We formulate the axioms and describe a number of models. Supersymmetry transformations are implemented as superunitary transformations in a space defined over a Grassmann algebra; generalised Bogoliubov transformations acting in that space are also formulated.

1. Introduction

Since the invention of supersymmetric field theoretical models [1], a steadily increasing literature exists on that subject and a number of reviews, books and conference reports are available [2-10]. The most striking aspect of supersymmetry concerns the fact that one relates bosonic degrees of freedom to fermionic ones. In other words, fields obeying the canonical commutation relations (CCR) are connected to fields obeying canonical anticommutation relations (CAR). Usually anticommuting parameters are introduced and transformations are formulated in an appropriate superspace. The main developments were connected to a study of field theoretical models, the most exciting result being the observation that there are models where no ultraviolet divergences show up.

The common treatment of bosons and fermions leads to a gradation of the underlying Hilbert space. The generators of supersymmetry mapping from even to odd states and vice versa are assumed to fulfil a Lie superalgebra. These algebraic aspects have been intensively studied since the early days. Soon after the invention of supersymmetry the most general possible generators for such symmetries of the S matrix were obtained [11]. Afterwards many people studied especially the structure of Lie superalgebras and their representations [12-17]. A complete classification of the simple Lie superalgebras was obtained [12]. The main questions, which are still not completely settled, concern the notion of supermanifolds [18] and Lie supergroups.

Besides the study of complicated systems with infinite degrees of freedom, a study of systems with a finite number of degrees of freedom may shed some light on supersymmetric models. Witten was the first to initiate such studies [19]; the problem of supersymmetry breaking for quantum mechanical models was extensively discussed

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in [20], and the strong restrictions imposed on Hamiltonians for supersymmetric quantum mechanics (SSQM) were worked out in [21]. Afterwards special properties of models were studied by a number of authors; let us mention only two recent contributions [22, 23].

We shall concentrate in this paper on the principal formulation of SSQM. In order to formulate quantum mechanics for f bosonic and f fermionic degrees of freedom, we work in the tensor product of the appropriate Hilbert spaces. A Klein-Jordan-Wigner transformation allows us to relate this space to a Hilbert space $\mathscr{H}_f$ built over a Grassmann algebra (§ 2). With the help of Grassmann variables we transform the CCR to the CAR. Bounded operators are obtained because we have to divide through by the square root of the partial Hamiltonians (§ 3).

We realise that an axiomatic framework of SSQM can be formulated in terms of sesquilinear forms rather than as operator equations restricted to an appropriate invariant common core as was proposed in [24].

Let $\mathscr{H}_f = \mathscr{H}^0 \oplus \mathscr{H}^1$ be the decomposition of the space $\mathscr{H}_f$ into an orthogonal sum of two infinite-dimensional subspaces $\mathscr{H}^0$ and $\mathscr{H}^1$; the elements of $\mathscr{H}^0$ and $\mathscr{H}^1$ are called even and odd, respectively. Denote by N_0 and $N_1 = 1 - N_0$ the corresponding projection operators onto $\mathscr{H}^0$ and $\mathscr{H}^1$ and introduce the Klein operator $K = N_0 - N_1$.

Axiom 1. Assume that there are given self-adjoint operators $Q^n = Q^{n\dagger}$, $n = 1, \ldots, N$, and a non-negative Hamiltonian H, such that

$$(Q^n)^2 = H, \qquad \operatorname{dom} Q^n = \operatorname{dom} H^{1/2} = \mathscr{D}, \qquad n = 1, \ldots, N. \tag{1.1}$$

Axiom 2. Assume that these observables fulfil the anticommutation relations

$$\langle Q^n \Psi | Q^m \Phi \rangle + \langle Q^m \Psi | Q^n \Phi \rangle = 0 \qquad n \neq m \qquad \Psi, \Phi \in \mathscr{D}. \tag{1.2}$$

Axiom 3. Assume that the observables Q^n, $n = 1, \ldots, N$, map states from $\mathscr{H}^0$ to states of $\mathscr{H}^1$ and vice versa

$$\langle Q^n \Psi | K \Phi \rangle + \langle K \Psi | Q^n \Phi \rangle = 0 \qquad \Psi, \Phi \in \mathscr{D}. \tag{1.3}$$

Therefore H is reduced by both $\mathscr{H}^0$ and $\mathscr{H}^1$ and

$$\langle H \Psi | K \Phi \rangle + \langle K \Psi | H \Phi \rangle = 0 \qquad \Psi, \Phi \in \operatorname{dom} \mathscr{H} \tag{1.4}$$

follows. If $\mathscr{D}$ is an invariant domain for K with $K\mathscr{D} \subseteq \mathscr{D}$, it follows that $K \cdot \operatorname{dom}(Q^n)^p \subseteq \operatorname{dom}(Q^n)^p$ for $p = 2, 3, \ldots$, and especially $K \cdot \operatorname{dom} H \subseteq \operatorname{dom} H$ holds.

The physical interpretation of SSQM is based on the fact that eigenstates of N_0 are called bosonic ones whereas eigenstates of N_1 are called fermionic; Q^n are called supercharges, H is a non-negative Hamiltonian and the Klein operator can be written as $K = (-1)^{N_1}$.

From an algebraic point of view one therefore studies representations of the Lie superalgebra S(N) in a separable Hilbert space $\mathscr{H}_f$ which is Z_2 graded due to the projection operators N_0 and N_1.

Spectral properties of H are derived easily from the above axioms. For the energy spectrum of any supersymmetric model one obtains

$$\sigma_p(H)\backslash\{0\} = \sigma_p(HN_0)\backslash\{0\} = \sigma_p(HN_1)\backslash\{0\}. \tag{1.5}$$

Thus the restrictions of H to the bosonic and fermionic subspaces are 'essentially isospectral' [25]: except for a possible eigenvalue zero the point spectra of HN_0 and HN_1 coincide and the dimensions of their corresponding eigenspaces coincide as well.

In order to be able to formulate supersymmetry transformations we introduce anticommuting parameters in § 5. To be consistent these parameters have to be introduced also in the scalar product for the wavefunctions and the operators. We remark that they play a quite different role compared to the Grassmann variables mentioned before. They are not integrated over in the scalar product. They may be taken outside a scalar product by using the Klein operator. This property has been noted previously by van Hove [26] and more recently in [27] where supersymmetric models at finite temperatures are discussed. In order to obtain properties of the supertrace, the Klein operator turns out to be useful [28].

Supersymmetry transformations are treated in § 6 and generalised Bogoliubov transformations mixing even and odd elements are dealt with in § 7. These transformations turn out to be superunitary. Finally we treat representations of the Lie superalgebras $S(N)$ for general N by self-adjoint operators in a Hilbert space in § 8.

We have given a rigorous and self-contained treatment of SSQM in this paper. The consequences for an infinite number of degrees of freedom are under investigation.

2. Grassmann algebra for f non-relativistic fermions

The Grassmann algebra G_f of polynomials in f anticommuting variables $\varepsilon_1, \ldots, \varepsilon_f$, over the field of complex numbers $\mathbb{C}$, is an associative superalgebra with unit I and general element

$$\xi = \sum_{1 \leq i_1 < \ldots < i_p \leq f} c_{i_1 \ldots i_p} \varepsilon_{i_1} \ldots \varepsilon_{i_p} + c_0 I \qquad \{\varepsilon_i, \varepsilon_j\} = 0 \qquad i, j = 1, \ldots, f \tag{2.1}$$

with $c_0, c_{i_1, \ldots, i_p} \in \mathbb{C}$. It follows that the dimension of G_f is 2^f [29].

An element $\xi \in G_f$ is called even (odd), if it is a linear combination of products of an even (odd) number of variables ε_k, $k = 1, \ldots, f$.

The derivative from the left with respect to ε_k is defined by linear extension of the operator $\partial_k \equiv \partial / \partial \varepsilon_k$ acting on monomials as

$$\partial_k \varepsilon_{i_1}, \ldots \varepsilon_{i_p} = \delta_{k i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_p} - \delta_{k i_2} \varepsilon_{i_1} \varepsilon_{i_3} \ldots \varepsilon_{i_p} + \ldots + (-1)^{p-1} \delta_{k i_p} \varepsilon_{i_1} \ldots \varepsilon_{i_{p-1}}. \tag{2.2}$$

This endomorphism of G_f is a graded derivative [15, 17] since

$$\partial_k (\xi \eta) = (\partial_k \xi) \eta + (-1)^{\deg \xi} \xi (\partial_k \eta) \qquad k = 1, \ldots, f \tag{2.3}$$

where $\deg \xi = 0$ if ξ is even and $\deg \xi = 1$ if ξ is odd.

The Grassmann algebra G'_f of polynomials in the anticommuting variables $\partial_1, \ldots, \partial_f$ can be combined with G_f and yields a Clifford algebra K_{2f} of polynomials in the $2f$ variables $\varepsilon_k, \partial_k$, $k = 1, \ldots, f$, which obey the anticommutation relations [29]

$$\{\varepsilon_i, \varepsilon_k\} = \{\partial_i, \partial_k\} = 0 \qquad \{\varepsilon_i, \partial_k\} = \delta_{ik} I \qquad i, k = 1, \ldots, f. \tag{2.4}$$

With ξ defined in (2.1) and η similarly with coefficients $d_{i_1 \ldots i_p}$ and d_0 we define a scalar product by

$$\langle \xi | \eta \rangle = \sum_{1 \leq i_1 < \ldots < i_p \leq f} c^*_{i_1 \ldots i_p} d_{i_1 \ldots i_p} + c^*_0 d_0. \tag{2.5}$$

G_f becomes thus a unitary space of dimension 2^f. Definition (2.5) implies the relation

$$\partial_k = \varepsilon_k^\dagger \qquad \partial_k + \varepsilon_k \quad \text{and} \quad \mathrm{i}(\partial_k - \varepsilon_k) \quad \text{are self-adjoint} \qquad k = 1, \ldots, f. \tag{2.6}$$

Replacing the complex coefficients by x-dependent wavefunctions leads to the separable Hilbert space $\mathscr{L}_f = L^2(\mathrm{d}^f x) \otimes G_f$ with elements Ψ and scalar product $\langle \Psi | \Phi \rangle$:

$$\Psi(x) = \psi_0(x) I + \sum_{1 \leq i_1 < \ldots < i_p \leq f} \psi_{i_1 \ldots i_p}(x) \varepsilon_{i_1} \ldots \varepsilon_{i_p} \qquad \psi_{i_1 \ldots i_p}, \psi_0 \in L^2(\mathrm{d}^f x)$$
$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}^f} \mathrm{d}^f x (\psi_0^*(x) \phi_0(x) + \ldots + \psi_{1\ldots f}^*(x) \phi_{1\ldots f}(x)). \tag{2.7}$$

The Hilbert space $\mathscr{L}_f$ is isomorphic to the f-fold tensor product of x-dependent Pauli spinors, since the isomorphism

$$\underbrace{(L^2(\mathrm{d}^1 x) \otimes \mathbb{C}^2) \otimes \ldots \otimes (L^2(\mathrm{d}^1 x) \otimes \mathbb{C}^2)}_{f \text{ times}} \simeq L^2(\mathrm{d}^f x) \otimes \mathbb{C}^{2^f} \simeq L^2(\mathrm{d}^f x) \otimes G_f \tag{2.8}$$

holds. This may be deduced by starting from Pauli spin matrices and using a Klein-Jordan-Wigner transformation [30]

$$W_k = \exp\left(\mathrm{i}\pi \sum_{i=1}^{k-1} \sigma_i^+ \sigma_i^- \right) = \prod_{i=1}^{k-1} (-\sigma_i^3) = W_k^\dagger = W_k^{-1} \qquad k = 2, \ldots, f \qquad W_1 = I$$
$$C_k = \sigma_k^- W_k = W_k \sigma_k^- \qquad C_k^\dagger = \sigma_k^+ W_k = W_k \sigma_k^+ \qquad k = 1, \ldots, f \tag{2.9}$$
$$\sigma_k^\pm = I_2 \otimes \ldots \otimes I_2 \otimes \underset{\substack{\uparrow \\ k}}{\sigma^\pm} \otimes I_2 \otimes \ldots \otimes I_2 \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and σ_k^3 defined similarly. These spin matrices fulfil the canonical anticommutation relations (CAR)

$$\{C_i, C_k\} = 0 \qquad \{C_i, C_k^\dagger\} = \delta_{ik} I \qquad i, k = 1, \ldots, f \tag{2.10}$$

and $C_k^\dagger C_k = \sigma_k^+ \sigma_k^- = (I + \sigma_k^3)/2$.

The isomorphism (2.8) is now defined such that

$$C_k \leftrightarrow \partial_k = \varepsilon_k^\dagger \qquad C_k^\dagger \leftrightarrow \varepsilon_k \qquad k = 1, \ldots, f. \tag{2.11}$$

The decomposition of G_f into orthogonal subspaces of even and odd elements determines a Z_2 grading of G_f. By (2.8) one obtains a grading of $\mathscr{L}_f = \mathscr{L}_f^0 \oplus \mathscr{L}_f^1$ with orthogonal projectors N_0 and $N_1 = I - N_0$ onto bosonic and fermionic subspaces, respectively. $K = N_0 - N_1 = I - 2N_1$ defines then the Klein operator [26, 27], which is used in the axiomatics of supersymmetric quantum mechanics (SSQM). By means of the spectral resolution of the fermionic projector N_1 one can also express $K = (-)^{N_1}$.

As an example we get for $f = 2$, $N_0 = I - \partial_1 \varepsilon_1 - \partial_2 \varepsilon_2 + 2\partial_1 \varepsilon_1 \partial_2 \varepsilon_2$.

$L^2(\mathrm{d}^f x)$ may be replaced by any separable Hilbert space $\mathscr{H}$, and $\mathscr{L}_f$ is then replaced by $\mathscr{H}_f = \mathscr{H} \otimes G_f = \mathscr{H}_f^0 \oplus \mathscr{H}_f^1$.

3. Fermionic oscillator

The linear operators

$$B_k = (x_k + \partial/\partial x_k)\sqrt{2} \qquad B_k^\dagger = (x_k - \partial/\partial x_k)/\sqrt{2} \qquad k = 1, \ldots, f \tag{3.1}$$

on $L^2(\mathrm{d}^f x)$ are closed on dom $B_k^\dagger$ = dom B_k = dom $x_k \cap$ dom p_k, with $p_k = -\mathrm{i}\partial/\partial x_k$. They fulfil the canonical commutation relations (CCR)

$$[B_i, B_k] = 0 \qquad [B_i, B_k^\dagger] = \delta_{ik} I \qquad i, k = 1, \ldots, f. \tag{3.2}$$

The bosonic energies, defined as form sums

$$B_k^\dagger B_k = \tfrac{1}{2}(p_k^2 \dot{+} x_k^2) - \tfrac{1}{2}I = B_k B_k^\dagger - I \qquad k = 1, \ldots, f \tag{3.3}$$

are self-adjoint on dom $B_k^\dagger B_k =$ dom $B_k B_k^\dagger$, with spectrum $\sigma(B_k^\dagger B_k) = \{0, 1, 2, \ldots\}$. Adding the spin-flip energies $C_k^\dagger C_k$ one obtains the Hamiltonian operator [21, 31] defined as a form sum in $\mathscr{L}_f$ [32–34]

$$H = \sum_{k=1}^{f} H_k \qquad H_k = B_k^\dagger B_k + C_k^\dagger C_k = \tfrac{1}{2}(p_k^2 \dot{+} x_k^2) + \tfrac{1}{2}\sigma_k^3 \geqslant 0. \tag{3.4}$$

The 'partial supercharges' $Q_{ki} = B_k C_i^\dagger$ with $Q_{ki}^2 = 0$ fulfil the Lie superalgebra

$$\{Q_{kk}, Q_{kk}^\dagger\} = H_k \qquad [Q_{kk}, H_k] = 0 \qquad k = 1, \ldots, f. \tag{3.5}$$

Their sum yields the supercharge

$$Q = \sqrt{2} \sum_{k=1}^{f} Q_{kk} = \sqrt{2} \sum_{k=1}^{f} B_k C_k^\dagger = (Q^1 + \mathrm{i}Q^2)/\sqrt{2} \tag{3.6}$$

and since $Q_{kk} + Q_{kk}^\dagger$ and $\mathrm{i}(Q_{kk} - Q_{kk}^\dagger)$ are self-adjoint on dom $B_k \otimes \mathbb{C}^{2^f}$ [25, 35], the supercharges Q^1 and Q^2 are self-adjoint on

$$\text{dom } H^{1/2} = \text{dom } Q^1 = \text{dom } Q^2 = \bigcap_{k=1}^{f} \text{dom } B_k \otimes \mathbb{C}^{2^f} \tag{3.7}$$

due to the criteria used in § 4.

These supercharges give a representation of the Lie superalgebra S(2) in $\mathscr{L}_f$:

$$(Q^1)^2 = (Q^2)^2 = H \qquad \{Q^1, Q^2\} = 0$$

or (3.8)

$$\{Q, Q^\dagger\} = 2H \qquad Q^2 = 0 \qquad \text{thus} \qquad [Q, H] = 0$$

where all relations hold in the sense of forms (see §§ 4 and 6).

Use of (2.8) and (2.11) shows that Q and $Q^\dagger$ (denoted by Q^*) transform from $\mathscr{H}_f^0$ to $\mathscr{H}_f^1$ and vice versa:

$$Q^*\Psi_0 = \Psi_1 \qquad Q^*\Psi_1 = \Psi_0 \qquad \Psi_0 \in \mathscr{H}_f^0 \qquad \Psi_1 \in \mathscr{H}_f^1 \tag{3.9}$$

and $[K, H] = 0$ since $\{K, Q^*\} = 0$, in the sense of forms. This last anticommutator, together with the Liesuperalgebra S(2), is significant for SSQM.

Using the notation $H_k = B_k^\dagger B_k + \varepsilon_k \partial_k \geqslant 0$ and the definition

$$B_k' = \varepsilon_k B_k / H_k^{1/2} \qquad [H_k, \varepsilon_k B_k] = 0 \qquad \| B_k' \| \leqslant 1 \qquad k = 1, \ldots, f \tag{3.10}$$

one may transform the CCR, (3.2), to the CAR

$$\{B_i', B_k'\} = 0 \qquad \{B_i', B_k'^\dagger\} = \delta_{ik} I \qquad i, k = 1, \ldots, f. \tag{3.11}$$

We note that the commutators in (3.2) and (3.10) should be understood in the sense that spectral projections of the operators involved commute.

From the uniqueness of representations of the CAR for finite degrees of freedom we deduce that the bounded operators B_k' can be decomposed orthogonally into matrices which are unitarily equivalent to σ^+, with respect to the spectral resolution of H_k.

In order to decribe the connection between the CAR and CCR for any representation of the superalgebra S(2) in terms of self-adjoint operators, one needs so-called anticommuting parameters (see § 5).

4. Generalisation of the bosonic observables

We used the bosonic operators B_k in order to construct a representation of the CCR and in order to get the Hamiltonian for the fermionic oscillator (3.4). More generally they may be replaced by closed operators G_k which are densely defined in $\mathscr{H}$ and yield a representation of S(2) by self-adjoint supercharges in the tensor product space $\mathscr{H}_f = \mathscr{H} \otimes G_f$. To start with we quote first the following lemma.

Lemma 1. Let G_1 and G_2 be linear operators which are closed and densely defined in $\mathscr{H}$. The tensor products $D_k = C_k^\dagger G_k + C_k G_k^\dagger$, $k = 1, 2$, on $\mathscr{D}_1 = \operatorname{dom} D_1 = (\operatorname{dom} G_1^\dagger \otimes \mathbb{C}^2) \oplus (\operatorname{dom} G_1 \otimes \mathbb{C}^2)$, $\mathscr{D}_2 = \operatorname{dom} D_2 = (\operatorname{dom} G_2^\dagger \oplus \operatorname{dom} G_2) \otimes \mathbb{C}^2$, are self-adjoint in $\mathscr{H} \otimes \mathbb{C}^4$. Here and in the following the isomorphism (2.8) is used without explicitly referring to it.

The proof of this lemma follows from the well known implication [25, 35]. Let A be closed and densely defined in H; the tensor product $\sigma^+ A + \sigma^- A^\dagger$ is then self-adjoint on $\operatorname{dom} A^\dagger \oplus \operatorname{dom} A$ in $\mathscr{H} \otimes \mathbb{C}^2$.

The generalisation of lemma 1 to the case of f fermions leads to a second lemma

Lemma 2. Let G_k, $k = 1, \ldots, f$, be closed and densely defined operators in $\mathscr{H}$. The operators

$$D_k = \varepsilon_k G_k + \partial_k G_k^\dagger \qquad \operatorname{dom} D_k = \mathscr{D}_k \qquad k = 1, \ldots, f \tag{4.1}$$

are self-adjoint on their natural domains in $\mathscr{H}_f$.

Restricting our attention, for the moment, again to the case of two fermions on the real line, leads to the following theorem.

Theorem 1. Let G_k and D_k, $k = 1, 2$, be defined as in lemma 1. Define the symmetric operator $Q^1 = D_1 + D_2$ on $\mathscr{D} = \operatorname{dom} Q^1 = \mathscr{D}_1 \cap \mathscr{D}_2$ and assume that

$$\langle G_1^\dagger \psi | G_2 \phi \rangle - \langle G_2^\dagger \psi | G_1 \phi \rangle = 0 \tag{4.2}$$

for $\phi \in \operatorname{dom} G_1 \cap \operatorname{dom} G_2$ and $\psi \in \operatorname{dom} G_1^\dagger \cap \operatorname{dom} G_2^\dagger$.

If there exists some positive $\alpha < 1$ such that the estimate

$$2\,\mathrm{Re}(\langle G_1^\dagger \psi | G_2^\dagger \phi \rangle - \langle G_2 \psi | G_1 \phi \rangle) + \| \psi \|^2 + \| \phi \|^2$$
$$+ \alpha(\| G_1^\dagger \psi \|^2 + \| G_2 \psi \|^2 + \| G_1 \phi \|^2 + \| G_2^\dagger \phi \|^2) \geq 0 \tag{4.3}$$

holds for $\psi \in \operatorname{dom} G_1^\dagger \cap \operatorname{dom} G_2$ and $\phi \in \operatorname{dom} G_1 \cap \operatorname{dom} G_2^\dagger$, then the operator sum Q^1 is closed.

Proof. One follows [36] where it is shown with A and B closed the sum $A + B$ is closed on $\operatorname{dom} A \cap \operatorname{dom} B$ if and only if there exists $\gamma > 0$ such that

$$\langle A\psi | A\psi \rangle + \langle B\psi | B\psi \rangle \leq \gamma \langle (A+B)\psi | (A+B)\psi \rangle + \gamma \| \psi \|^2 \tag{4.4}$$

holds for $\psi \in \operatorname{dom} A \cap \operatorname{dom} B$. Denoting $A = D_1$, $B = D_2$ and $1 - 1/\gamma = \alpha < 1$ thus proves theorem 1.

Remark. In order to establish that Q^1 has the properties of a supercharge we need further assumptions. Let G_k and D_k, $k=1,2$, be defined as in lemma 1 and Q^1 as in theorem 1. The self-adjoint non-negative Hamilton operator

$$H = Q^{1+}\overline{Q^1} \text{ on dom } H = \{\Psi \in \text{dom } \overline{Q^1}, \overline{Q^1}\Psi \in \text{dom } Q^{1+}\} \tag{4.5}$$

on the form domain dom $H^{1/2} =$ dom $\overline{Q^1}$ is an extension of $(\overline{Q^1})^2$. Here $\overline{Q^1}$ denotes the closure of the symmetric operator Q^1. If dom $(\overline{Q^1})^2$ is a core for H, Q^1 is essentially self-adjoint in $\mathscr{H}\otimes\mathbb{C}^4$ [37]. In order to verify that $(Q^1)^2$ is essentially self-adjoint on some suitable core for H, one may use Wüst's theorem [33, 34].

Corollary 1. Under the assumptions of theorem 1 similar conclusions hold for the symmetric operator

$$Q^2 = \mathrm{i} \sum_{k=1,2} (C_k G_k^+ - C_k^+ G_k) \qquad \text{dom } Q^2 = \text{dom } Q^1 = \text{dom } H^{1/2}. \tag{4.6}$$

Moreover, one obtains the algebra

$$Q^{1+}Q^1 = Q^{2+}Q^2 = H \qquad \langle Q^1\Psi | Q^2\Phi\rangle + \langle Q^2\Psi | Q^1\Phi\rangle = 0 \qquad \Phi, \Psi \in \mathscr{D}. \tag{4.7}$$

If dom $(\overline{Q^2})^2$ is a core for H, Q^2 is self-adjoint.

Remark. Under the assumptions of theorem 1 it follows that we obtain a representation of the Lie superalgebra S(2) in $\mathscr{H}\otimes\mathbb{C}^4$:

$$\begin{aligned} &(Q^1)^2 = (Q^2)^2 = H \qquad Q^1 = Q^{1+} \qquad Q^2 = Q^{2+} \\ &\langle Q^1\Psi | Q^2\Phi\rangle + \langle Q^2\Psi | Q^1\Phi\rangle = 0 \qquad \Phi, \Psi \in \text{dom } Q^1 = \text{dom } Q^2 = \text{dom } H^{1/2}. \end{aligned} \tag{4.8}$$

A special model of SSQM in two dimensions is obtained by choosing $A(x_1, x_2)$ and $V(x_1, x_2)$ as absolutely continuous functions of x_1, $x_2 \in \mathbb{R}$, and then defining

$$\begin{aligned} &G_k = (\mathrm{i}\pi_k + V_k)/\sqrt{2} \qquad \pi_k = p_k + A_k \qquad A_k = \frac{\partial}{\partial x_k} A(x_1, x_2) \\ &V_k = \frac{\partial}{\partial x_k} V(x_1, x_2) \qquad k = 1, 2. \end{aligned} \tag{4.9}$$

Again (4.4) may be used to deduce that G_k is closed on dom $G_k =$ dom $p_k \cap$ dom $A_k \cap$ dom V_k if and only if there exists a positive $\alpha < 1$ such that

$$\alpha\left(-\frac{\partial^2}{\partial x^2} + V_k^2 + A_k^2\right) + I + (V_k - \mathrm{i}A_k)\frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_k}(V_k + \mathrm{i}A_k) \geqslant 0 \qquad k = 1, 2 \tag{4.10}$$

in the sense of forms on dom G_k.

If A_k is relatively bounded with respect to p_k with relative bound less than one such that π_k is self-adjoint on dom $\pi_k =$ dom p_k, we may define

$$\begin{aligned} &H_k' = \pi_k^2 \dot{+} V_k^2 = \{G_k, G_k^+\} \\ &V_{ik} = \frac{\partial}{\partial x_i} V_k - V_k \frac{\partial}{\partial x_i} = [G_i, G_k^+] = [G_k, G_i^+] \qquad i, k = 1, 2 \end{aligned} \tag{4.11}$$

in the sense of form sums, form dom $H_k' =$ form dom $V_{ik} =$ dom $p_k \cap$ dom $V_k =$ dom $G_k \subseteq$ dom G_k^+, and $G_k^+|_{\text{dom}\, G_k} = (-\mathrm{i}\pi_k + V_k)/\sqrt{2}$. Then inequality (4.3) follows from the two estimates

$$|V_{12}| \leqslant I - (\alpha/2)(H_1' \pm V_{11} + H_2' \mp V_{22}) \tag{4.12}$$

for some positive $\alpha < 1$. Thus Q^1 is closed if (4.12) holds in the sense of forms on $\mathrm{dom}\, p_1 \cap \mathrm{dom}\, V_1 \cap \mathrm{dom}\, p_2 \cap \mathrm{dom}\, V_2$.

Wüst's theorem [33, 34] may be used to prove essential self-adjointness of Q^1 as well. Take $A_1 = A_2 = 0$ and assume that $V_k \in L^4_{\mathrm{loc}}(\mathbb{R}^2)$, $k = 1, 2$, such that $V_1^2 + V_2^2 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$; then

$$\dot{H}'_0 = -\frac{\partial^2}{\partial x_1^2} + V^2 - \frac{\partial^2}{\partial x_2^2} + V_2^2|_{C_0^\infty(\mathbb{R}^2)} = (H'_1 \dotplus H'_2)|_{C_0^\infty(\mathbb{R}^2)} \tag{4.13}$$

is essentially self-adjoint [33]. If the four estimates

$$\|(\pm V_{11} \pm V_{22})\psi\| \leqslant \gamma \|H'_0\psi\| + \beta\|\psi\| \qquad \text{for all } \psi \in \mathrm{dom}\, H'_0 \tag{4.14}$$

hold with $0 < \gamma \leqslant 1, \beta > 0$, where $H'_0 = H'_1 \dotplus H'_2$ is the unique self-adjoint extension of $\dot{H}'_0$, then

$$H_0 = D_1^2 \dotplus D_2^2 = \bigoplus_{k=1}^{4} H_0^k \qquad \text{on} \qquad \mathrm{dom}\, H_0^{1/2} = \mathrm{dom}\, Q^1 \tag{4.15}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes \mathbb{C}^4$. If, moreover, the two estimates

$$\|V_{12}\psi\|^2 \leqslant \beta\|\psi\|^2 + (\gamma/4)\|H_0^k\psi\|^2 \qquad 0 < \gamma \leqslant 1, \beta > 0 \tag{4.16}$$

hold for all $\psi \in \mathrm{dom}\, H_0^k$, $k = 2, 3$, then $(Q^1)^2|_{C_0^\infty(\mathbb{R}^2)\otimes\mathbb{C}^4}$ is essentially self-adjoint. In this case Q^1 is also essentially self-adjoint [37]. If the relative bounds γ in both estimates (4.14) and (4.16) are less than one, both H_0 and $\overline{(Q^1)^2} = (\overline{Q^1})^2$ are self-adjoint on $\mathrm{dom}\,(H'_1 \dotplus H'_2)$ [32-34]. Therefore if (4.12) is fulfilled too, self-adjoint supercharges Q^1 and Q^2 exist.

For more than two fermions, $f = 3, 4, \ldots$, proposition (4.4) cannot be used. Instead of (4.4) one may apply the KLMN theorem [32, 33]. One considers the symmetric operator

$$Q^1 = \sum_{k=1}^{f} D_k \qquad \text{on} \qquad \mathrm{dom}\, Q^1 = \bigcap_{k=1}^{f} \mathrm{dom}\, D_k = \mathscr{D} \tag{4.17}$$

where D_k are the closed densely defined operators of (4.1). The form sum

$$\sum_{k=1}^{f} \langle D_k\Psi | D_k\Phi\rangle = \langle D\Psi | D\Phi\rangle \qquad D = D^\dagger \qquad \Phi, \Psi \in \mathscr{D} \tag{4.18}$$

defines the non-negative operator D on $\mathrm{dom}\, D = \mathscr{D}$. If there exists a positive $\alpha < 1$ such that

$$\left|\sum_{i \neq k} \langle D_i\Psi | D_k\Psi\rangle\right| \leqslant \alpha\langle D\Psi | D\Psi\rangle \qquad \text{for all } \Psi \in \mathscr{D} \tag{4.19}$$

then there exists a non-negative operator $H = H^\dagger$ on $\mathrm{dom}\, H^{1/2} = \mathscr{D}$, such that for Φ, $\Psi \in \mathscr{D}$

$$\langle H^{1/2}\Psi | H^{1/2}\Phi\rangle = \langle D\Psi | D\Phi\rangle + \sum_{i \neq k} \langle D_i\psi | D_k\Phi\rangle = \langle Q^1\Psi | Q^1\Phi\rangle \tag{4.20}$$

holds. Therefore Q^1 is closed and $H = Q^{1\dagger}Q^1$.

Theorem 2. Let D_k, $k = 1, \ldots, f$, and Q^1 be defined as in (4.1) and (4.17). Assume that

$$\begin{gathered} \langle G_i^\dagger\psi | G_k\phi\rangle = \langle G_k^\dagger\psi | G_i\phi\rangle \\ \phi \in \mathrm{dom}\, G_i \cap \mathrm{dom}\, G_k \qquad \psi \in \mathrm{dom}\, G_i^\dagger \cap \mathrm{dom}\, G_k^\dagger \end{gathered} \tag{4.21}$$

holds for $i, k = 1, \ldots, f$. If

$$\left| 2\mathrm{Re} \sum_{i<k} (\langle G_i^\dagger \partial_i \Psi | G_k^\dagger \partial_k \Psi\rangle - \langle G_k \partial_i \Psi | G_i \partial_k \Psi\rangle) \right| \leqslant \alpha \sum_{k=1}^{f} (\| \partial_k G_k^\dagger \Psi \|^2 + \| \varepsilon_k G_k \Psi \|^2) \tag{4.22}$$

with $0 < \alpha < 1$, then Q^1 is closed. If, furthermore, $\mathrm{dom}(Q^1)^2$ is a core for $H = Q^{1\dagger} Q^1$, then Q^1 is self-adjoint in $\mathscr{H}_f$.

Corollary 2. Under the assumptions of theorem 2, the same conclusions hold for the operator

$$Q^2 = \mathrm{i} \sum_{k=1}^{f} (\partial_k G_k^\dagger - \varepsilon_k G_k) \qquad \text{on} \qquad \mathrm{dom}\, Q^2 = \mathrm{dom}\, Q^1 = \mathrm{dom}\, H^{1/2} \tag{4.23}$$

and the algebra (4.7) holds on the form domain $\mathrm{dom}\, H^{1/2}$. If, especially, both $\mathrm{dom}\,(Q^1)^2$ and $\mathrm{dom}\,(Q^2)^2$ are cores for H, then the self-adjoint supercharges Q^1, Q^2 fulfil the following self-adjoint representation of the Lie superalgebra S(2) in $\mathscr{H}_f$:

$$(Q^1)^2 = (Q^2)^2 = H \qquad \langle Q^1\Psi | Q^2\Phi\rangle + \langle Q^2\Psi | Q^1\Phi\rangle = 0 \qquad \Phi, \Psi \in \mathrm{dom}\, H^{1/2}. \tag{4.24}$$

In addition, the Klein operator $K = N_0 - N_1$ fulfils the anticommutation relations

$$\langle K\Psi | Q^k\Phi\rangle + \langle Q^k\Psi | K\Phi\rangle = 0 \qquad \text{for} \qquad \Phi, \Psi \in \mathrm{dom}\, H^{1/2} \qquad k = 1, 2 \tag{4.25}$$

which imply that

$$\langle K\Psi | H\Phi\rangle - \langle H\Psi | K\Phi\rangle = 0 \qquad \text{for} \qquad \Phi, \Psi \in \mathrm{dom}\, H. \tag{4.26}$$

Let the differential operators G_k be densely defined, generalising (4.9) with locally absolutely continuous superpotentials $V(x_1, \ldots x_f)$ and $A(x_1, \ldots, x_f)$ obeying (4.10), $k = 1, \ldots f$; assume that A_k is relatively bounded with respect to p_k with relative bound less than one. The inequality (4.22) follows from the estimates

$$\sum_{\substack{i=1 \\ i \neq k}}^{f} | V_{ik} | \leqslant \tfrac{1}{2}\alpha (H'_k + V_{kk}) \qquad k = 1, \ldots, f \qquad 0 < \alpha < 1 \tag{4.27}$$

in the sense of forms on $\bigcap_{k=1}^{f} \mathrm{dom}\, p_k \cap \mathrm{dom}\, V_k$. The corresponding Hamilton operator in $\mathscr{L}_f$ is given by

$$H = \tfrac{1}{2} \sum_{k=1}^{f} (\pi_k^2 \dot{+} V_k^2 + V_{kk}\sigma_k^3) + \sum_{i \neq k} V_{ki}\varepsilon_k \partial_i \geqslant 0 \tag{4.28}$$

on

$$\mathrm{dom}\, H^{1/2} = \mathrm{dom}\, Q^1 = \mathrm{dom}\, Q^2 = \bigcap_{k=1}^{f} \mathrm{dom}\, D_k = \bigcap_{k=1}^{f} \mathrm{dom}\, G_k \otimes G_f$$

with

$$\mathrm{dom}\, G_k = \mathrm{dom}\, p_k \cap \mathrm{dom}\, V_k.$$

In order to investigate whether Q^1 is essentially self-adjoint one may again proceed in two steps. Assume that $V_k \in L^4_{\mathrm{loc}}(\mathbb{R}^f)$, $k = 1, \ldots, f$, such that

$$H'_0 \Big|_{C_0^\infty(\mathbb{R}^f)} = \sum_{k=1}^{f} \left(-\frac{\partial^2}{\partial x_k^2} + V_k^2 \right) \Bigg|_{C_0^\infty(\mathbb{R}^f)} \qquad H'_0 = H'_1 \dot{+} \ldots \dot{+} H'_f \tag{4.29}$$

is essentially self-adjoint. The estimates (4.14) can be generalised to the 2^f inequalities:

$$\|(\pm V_{11}\pm\ldots\pm V_{ff})\psi\|\leq\gamma\|H'_0\psi\|+\beta\|\psi\| \qquad \text{for all } \psi\in\operatorname{dom} H'_0$$
$$0<\gamma\leq 1 \qquad \beta>0. \tag{4.30}$$

This implies that $H_0=D_1^2\dot{+}\ldots\dot{+}D_f^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^f)\otimes G_f$. A suitable set of 2^f-2 sufficient conditions in order to have $(Q^1)^2$ essentially self-adjoint on $C_0^\infty(\mathbb{R}^f)\otimes G_f$ is given by

$$\sum_{\substack{i\neq k_1\ldots k_r\\ 1\leq k_1<\ldots<k_r\leq f}}(\|V_{ik_1}\psi\|^2+\ldots+\|V_{ik_r}\psi\|^2)$$
$$\leq\beta\|\psi\|^2+(\gamma/4)\Big\|\sum_{i\neq k_1\ldots k_r}(H'_i-V_{ii})\psi\dot{+}\sum_{i=k_1\ldots k_r}(H'_i+V_{ii})\psi\Big\|^2 \tag{4.31}$$
$$r=1,\ldots,f-1 \qquad 0<\gamma\leq 1 \qquad \beta>0$$

with the components ψ of $\Psi\in\operatorname{dom} H_0$. If γ in (4.30) and (4.31) are less than one, then both H_0 and $(Q^1)^2=(Q^1)^2$ are self-adjoint on the domain

$$\operatorname{dom} H_0=\operatorname{dom}\left[\left(-\frac{\partial^2}{\partial x_1^2}\right)\dot{+}V_1^2\dot{+}\ldots\dot{+}\left(-\frac{\partial^2}{\partial x_f^2}\right)\dot{+}V_f^2\right]\otimes G_f.$$

The automorphism of the superalgebra (4.24) which is generated by the third component Σ of the total spin of f fermions

$$\left.\begin{array}{l}\langle\Sigma\Phi|Q^1\Psi\rangle-\langle Q^1\Phi|\Sigma\Psi\rangle=\mathrm{i}\langle\Phi|Q^2\Psi\rangle\\ \langle\Sigma\Phi|Q^2\Psi\rangle-\langle Q^2\Phi|\Sigma\Psi\rangle=-\mathrm{i}\langle\Phi|Q^1\Psi\rangle\end{array}\right\}\Phi,\Psi\in\operatorname{dom} H^{1/2} \tag{4.32}$$

gives an integral of motion

$$\langle\Sigma\Phi|H\Psi\rangle-\langle H\Phi|\Sigma\Psi\rangle=0 \qquad \Phi,\Psi\in\operatorname{dom} H. \tag{4.33}$$

The 'partial supercharges'

$$Q_k^1=\varepsilon_kG_k+\partial_kG_k^\dagger=D_k \qquad Q_k^2=\mathrm{i}(\partial_kG_k^\dagger-\varepsilon_kG_k) \tag{4.34}$$

together with the Hamiltonians

$$H_k=\tfrac{1}{2}(\pi_k^2\dot{+}V_k^2+V_{kk}\sigma_k^3)=G_k^\dagger G_k+V_{kk}\varepsilon_k\partial_k\geq 0 \tag{4.35}$$

satisfy the Lie superalgebra S(2) for each $k=1,\ldots,f$. If especially $V_{ik}=\delta_{ik}V_{kk}$, $i,k=1,\ldots,f$, then the $3f$ operators Q_k^1, Q_k^2 and H_k, $k=1,\ldots,f$, generate a Lie superalgebra with the only non-zero supercommutators $\{Q_k^i,Q_k^i\}=2H_k$, $i=1,2$, and in this case the form sum

$$\sum_{k=1}^{f}H_k=H$$

gives the total Hamiltonian.

5. Anticommuting parameters

We introduce the Grassmann algebra $\mathscr{D}_2$ of polynomials in the anticommuting variables

Θ_1 and Θ_2 with complex coefficients and define an involution by the following rules:

$$\begin{aligned}&\Theta_1^2=\Theta_2^2=0 \qquad \Theta_1\Theta_2+\Theta_2\Theta_1=0 \qquad \Theta_1^*=\Theta_1 \qquad \Theta_2^*=\Theta_2\\ &(\Theta_1\Theta_2)^*=\Theta_2\Theta_1 \qquad \Theta=\Theta_1+\mathrm{i}\Theta_2 \qquad \Theta^*=\Theta_1-\mathrm{i}\Theta_2\\ &\Theta^2=\Theta^{*2}=0 \qquad \Theta^{**}=\Theta \qquad \Theta^*\Theta=2\mathrm{i}\Theta_1\Theta_2\\ &(c_0+c_1\Theta_1+c_2\Theta_2+c_{12}\Theta_1\Theta_2)^*=c_0^*+c_1^*\Theta_1+c_2^*\Theta_2+c_{12}^*\Theta_2\Theta_1.\end{aligned} \tag{5.1}$$

The tensor product of this Grassmann algebra $\mathscr{D}_2$ with the Clifford algebra K_{2f} of polynomials in ε_k and ∂_k, $k=1,\dots,f$, is an associative superalgebra over $\mathbb{C}$.

The Hilbert space $\mathscr{H}_f$ is extended to a $\mathscr{D}_2$-modul [38] with general element

$$\Psi=\Psi^0+\Theta_1\Psi^1+\Theta_2\Psi^2+\Theta_1\Theta_2\Psi^{12} \qquad \Psi^0,\Psi^1,\Psi^2,\Psi^{12}\in\mathscr{H}_f. \tag{5.2}$$

These anticommuting parameters are also introduced into the scalar product according to the following rules:

$$\begin{aligned}&\langle\Theta_j\phi\,|\,\psi\rangle=\langle\phi\,|\,\Theta_j\psi\rangle=\Theta_j\langle\phi\,|\,\psi\rangle \qquad j=1,2\\ &\langle\Theta_2\Theta_1\phi\,|\,\psi\rangle=\langle\Theta_1\phi\,|\,\Theta_2\psi\rangle=\langle\phi\,|\,\Theta_1\Theta_2\psi\rangle=\Theta_1\Theta_2\langle\phi\,|\,\psi\rangle\\ &\langle\Theta^*\phi\,|\,\psi\rangle=\langle\phi\,|\,\Theta\psi\rangle=\Theta\langle\phi\,|\,\psi\rangle \qquad \phi,\psi\in\mathscr{H}.\end{aligned} \tag{5.3}$$

In addition, one demands the relations [26, 27]

$$\begin{aligned}&(\varepsilon_k\Theta_j)^\dagger=\Theta_j\partial_k \qquad (\varepsilon_k\Theta)^\dagger=\Theta^*\partial_k \qquad j=1,2\\ &\langle\varepsilon_k\phi\,|\,\varepsilon_k\Theta\psi\rangle=\langle\Theta^*\partial_k\varepsilon_k\phi\,|\,\psi\rangle=\langle\Theta^*\phi\,|\,\psi\rangle=\Theta\langle\phi\,|\,\psi\rangle \qquad k=1,\dots,f\end{aligned} \tag{5.4}$$

for ϕ, $\psi\in\mathscr{H}$; more generally we require that

$$\begin{aligned}&\langle\Phi\,|\,(c_1\Theta_1+c_2\Theta_2)\Psi\rangle=\langle(c_1^*\Theta_1+c_2^*\Theta_2)\Phi\,|\,\Psi\rangle=(c_1\Theta_1+c_2\Theta_2)\langle\Phi\,|\,K\Psi\rangle\\ &\langle\Phi\,|\,\Theta_1\Theta_2\Psi\rangle=\Theta_1\Theta_2\langle\Phi\,|\,\Psi\rangle \qquad \text{for} \qquad \Phi,\Psi\in\mathscr{H}_f\end{aligned} \tag{5.5}$$

holds. Note that there is no integration over Θ_i included.

We illustrate these rules by the following examples:

$$\langle\varepsilon_1\phi\,|\,\Theta_1\varepsilon_1\psi\rangle=\langle\phi\,|\,\partial_1\Theta_1\varepsilon_1\psi\rangle=-\langle\phi\,|\,\Theta_1\partial_1\varepsilon_1\psi\rangle=-\langle\phi\,|\,\Theta_1\psi\rangle=-\Theta_1\langle\phi\,|\,\psi\rangle \tag{5.6}$$

$$\langle\varepsilon_1\Theta^*\phi\,|\,\Theta_2\varepsilon_1\psi\rangle=\langle\Theta^*\phi\,|\,\partial_1\Theta_2\varepsilon_1\psi\rangle=-\langle\Theta^*\phi\,|\,\Theta_2\psi\rangle=-\langle\phi\,|\,\Theta\Theta_2\psi\rangle=\Theta_2\Theta_1\langle\phi\,|\,\psi\rangle \tag{5.7}$$

for ϕ, $\psi\in\mathscr{H}$.

Linear operators in $\mathscr{H}_f$ are extended too, since linear combinations with anticommuting parameters Θ_1, Θ_2, $\partial/\partial\Theta_1$ and $\partial/\partial\Theta_2$ as coefficients may occur. The derivatives from the left $\partial/\partial\Theta_1$ and $\partial/\partial\Theta_2$ are to be understood as odd endomorphism of the Grassmann algebra $\mathscr{D}_2$, similarly as in (2.2), but are *not* the 'adjoints' of Θ_1 and Θ_2.

To be consistent one has to require that the left derivatives $\partial/\partial\Theta_j$, $j=1,2$, anticommute both with the fermionic operators ε_k and ∂_k, $k=1,\dots,f$.

The notion of trace may be generalised in two different ways (cf [28]). Start with an orthonormal basis $\{\Phi_n;\, n\in N\}$ for $\mathscr{H}_f$, with grading $\beta_n=0$ or 1 if $\Phi_n\in\mathscr{H}_f^0$ or $\mathscr{H}_f^1$, respectively. Assume that the operators T, T_1, T_2, T_{12} are trace class in $\mathscr{H}_f$ [32, 34]. We then define a supertrace and a trace by

$$\begin{aligned}\operatorname{str}(T+\Theta_1T_1+\Theta_2T_2+\Theta_1\Theta_2T_{12})&=\sum_{n=1}^{\infty}(-1)^{\beta_n}\langle\Phi_n\,|\,(T+\Theta_1T_1+\Theta_2T_2+\Theta_1\Theta_2T_{12})\Phi_n\rangle\\ &=\operatorname{str}T+\Theta_1\operatorname{tr}T_1+\Theta_2\operatorname{tr}T_2+\Theta_1\Theta_2\operatorname{str}T_{12}\end{aligned} \tag{5.8}$$

$$\mathrm{tr}(T+\Theta_1 T_1+\Theta_2 T_2+\Theta_1\Theta_2 T_{12})=\sum_{n=1}^{\infty}\langle\Phi_n|(T+\Theta_1 T_1+\Theta_2 T_2+\Theta_1\Theta_2 T_{12})\Phi_n\rangle$$
$$=\mathrm{tr}\,T+\Theta_1\,\mathrm{str}\,T_1+\Theta_2\,\mathrm{str}\,T_2+\Theta_1\Theta_2\,\mathrm{tr}\,T_{12}.$$

One therefore gets, for instance,

$$\mathrm{tr}(\Theta_j T_j)=\sum_{n=1}^{\infty}\Theta_j\langle\Phi_n|KT_j\Phi_n\rangle=\Theta_j\,\mathrm{str}\,T_j \qquad \text{for } j=1,2. \tag{5.9}$$

One easily verifies the following rules: tr T = str $T=0$ if T is an odd endomorphism of $\mathscr{H}_f$ and trace class; str $\{S, T\}=0$ for Hilbert–Schmidt operators S and T if at least one of them is odd; str $[S, T]=0$ for even Hilbert–Schmidt operators S, T.

Furthermore, one obtains

$$\mathrm{str}\,[\Theta_1 T_1, \Theta_2 T_2]=\Theta_2\Theta_1\,\mathrm{str}\,\{T_1, T_2\}=0 \tag{5.10}$$

for odd Hilbert–Schmidt operators T_1, T_2

$$\mathrm{str}\{\Theta_j T_j, S\}=\Theta_j\,\mathrm{tr}[T_j, S]=0 \tag{5.11}$$

for Hilbert–Schmidt operators S, T_j, $j=1,2$, if S is odd. These rules are easily derived from the relations

$$\mathrm{str}\,T=\mathrm{tr}(KT) \qquad \mathrm{tr}\,T=\mathrm{str}(KT) \tag{5.12}$$

with T being trace class on $\mathscr{H}_f$.

6. Supersymmetry transformations

In order to define an appropriate group of supersymmetry transformations [39–42] one has to convert the anticommutators of the fermionic operators, like the supercharges, to commutators using anticommuting parameters [3, 39]. We introduce the supercharges Q and $Q^\dagger$

$$\begin{aligned} &Q=(Q^1+\mathrm{i}Q^2)/\sqrt{2} && \mathrm{dom}\,Q=\mathrm{dom}\,Q^1=\mathrm{dom}\,Q^2=\mathscr{D} \\ &Q^\dagger\supseteq(Q^1-\mathrm{i}Q^2)/\sqrt{2} && \mathrm{dom}\,Q^\dagger\supseteq\mathrm{dom}\,Q \qquad Q^{\dagger\dagger}=\bar{Q} \end{aligned} \tag{6.1}$$

as linear combinations of the self-adjoint supercharges Q^1 and Q^2. The Lie superalgebra (4.24) is now equivalently written as

$$\begin{aligned} &\langle Q^\dagger\Psi|Q^\dagger\Phi\rangle+\langle Q\Psi|Q\Phi\rangle=2\langle H^{1/2}\Psi|H^{1/2}\Phi\rangle \\ &\langle Q^\dagger\Psi|Q\Phi\rangle=0 \qquad \Phi,\Psi\in\mathrm{dom}\,H^{1/2}=\mathrm{dom}\,Q \\ &\langle Q^\dagger\Psi'|H\Phi'\rangle=\langle H\Psi'|Q\Phi'\rangle \qquad \Phi',\Psi'\in\mathrm{dom}\,H. \end{aligned} \tag{6.2}$$

This can be transformed to Lie algebra relations using the anticommuting parameters defined above. In terms of forms we get

$$\begin{aligned} &\langle\Theta^* Q^\dagger\Psi|\Theta^* Q^\dagger\Phi\rangle-\langle Q\Theta\Psi|Q\Theta\Phi\rangle=2\Theta\Theta^*\langle H^{1/2}\Psi|H^{1/2}\Phi\rangle \\ &\langle\Theta^* Q^\dagger\Psi|Q\Theta\Phi\rangle=0 \qquad \Phi,\Psi\in\mathrm{dom}\,Q \\ &\langle\Theta^* Q^\dagger\Psi'|H\Phi'\rangle=\langle H\Psi'|Q\Theta\Phi'\rangle \qquad \Phi',\Psi'\in\mathrm{dom}\,H. \end{aligned} \tag{6.3}$$

Supersymmetry transformations are now defined as sesquilinear forms on dom Q by

$$\begin{aligned} g(t, s, r) &= \exp(\mathrm{i}tH + \mathrm{i}sQ\Theta + \mathrm{i}s^*\Theta^*Q^\dagger + \mathrm{i}rH\Theta\Theta^*) \\ &= \exp(\mathrm{i}tH)\exp(\mathrm{i}rH\Theta\Theta^*)\exp(\mathrm{i}sQ\Theta + \mathrm{i}s^*\Theta^*Q^\dagger) \\ &= \exp(\mathrm{i}tH)(I + \mathrm{i}rH\Theta\Theta^* + \mathrm{i}sQ\Theta + \mathrm{i}s^*\Theta^*Q^\dagger - \tfrac{1}{2}|s|^2\{Q\Theta, \Theta^*Q^\dagger\}) \end{aligned} \qquad (6.4)$$

for t and r real, and s complex. The composition law in terms of these parameters becomes

$$g(t, s, r)g(t', s', r') = g(t'', s'', r'')$$

with (6.5)

$$t'' = t + t' \qquad s'' = s + s' \qquad r'' = r + r' + 2\,\mathrm{Im}(s' \cdot s^*)$$

These transformations are 'superunitary' in the following sense:

$$g(t, s, r)^\dagger = g(t, s, r)^{-1} = g(-t, -s, -r) \qquad t, r \in \mathbb{R} \qquad s \in \mathbb{C}. \qquad (6.6)$$

Instead of supercharges from (4.1) we may insert the partial supercharges $Q_k = (Q_k^1 + \mathrm{i}Q_k^2)/\sqrt{2}$ defined in § 4.

In order to formulate the transformations generated by the Lie superalgebra $\{Q_k, Q_k^\dagger, H_k;\ k = 1, \ldots, f\}$ for the case with $V_{ki} = 0$ for $k \neq i$, $i, k = 1, \ldots, f$, one needs anticommuting parameters $\Theta^k = \Theta_1^k + \mathrm{i}\Theta_2^k$ with $(\Theta_j^k)^* = \Theta_j^k$ for $j = 1, 2$, $k = 1, \ldots, f$; in that case one obtains the direct product of f supergroups of type (6.4).

We remark on one problem. The exponentials of expressions with anticommuting parameters are defined as power series and in fact become polynomials. Therefore, superunitary transformations from above need not be defined on the whole of $\mathscr{H}_f$, for example,

$$\Psi(t, s, r) = g(t, s, r)\Psi_0 = \exp(\mathrm{i}tH)\Psi_0 + \Theta_1\Psi^1 + \Theta_2\Psi^2 + \Theta_1\Theta_2\Psi^{12} \qquad (6.7)$$

is defined for any $\Psi_0 \in \mathscr{H}_f$, but with Ψ^1, Ψ^2, Ψ^{12} only in the sense of distributions. But sesquilinear forms like, for instance,

$$\langle H\Phi' | \Theta\Theta^*\Psi \rangle = \Theta\Theta^*\langle H\Phi' | \Psi \rangle \qquad \langle Q\Phi | \Theta\Psi \rangle = \Theta\langle Q\Phi | K\Psi \rangle \qquad (6.8)$$

with $\Psi \in \mathscr{H}_f$, $\Phi \in \mathrm{dom}\, Q$, $\Phi' \in \mathrm{dom}\, H$, are well defined.

We notice that, although both ε_k and Θ_j are anticommuting elements, they play quite different roles within the framework of SSQM: whereas ε_k are bounded operators on $\mathscr{H}_f$ with adjoints $\varepsilon_k^\dagger = \partial_k$ and allow us to distinguish between bosons and fermions, Θ_1 and Θ_2 are used only as additional coefficients. The latter not only allow us to define supergroup transformations, as indicated above, but enable us also the define generalised Bogoliubov transformations which mix elements of CCR and CAR.

7. Generalised Bogoliubov transformation

We consider again the CCR (3.2) and the CAR (2.10) and use the isomorphism (2.11)

$$\begin{aligned} &[B_i, B_k] = 0 \qquad [B_i, B_k^\dagger] = \delta_{ik}I \qquad [B_i, \varepsilon_k] = [B_i^\dagger, \varepsilon_k] = 0 \\ &\{\varepsilon_i, \varepsilon_k\} = 0 \qquad \{\varepsilon_i, \varepsilon_k^\dagger\} = \delta_{ik}I \qquad i, k = 1, \ldots, f. \end{aligned} \qquad (7.1)$$

Next we introduce a Grassmann algebra $\mathscr{D} = \mathscr{D}_0 \oplus \mathscr{D}_1$ of polynomials in the anticommuting variables $\delta_1, \ldots, \delta_d$ over $\mathbb{C}$, and assume these variables to anticommute both with ε_k and $\varepsilon_k^\dagger = \partial_k$ for $k=1,\ldots,f$. $\mathscr{D}_0$ and $\mathscr{D}_1$ denote the even and odd subspace of the Lie superalgebra $\mathscr{D}$. Now we can define linear transformations of the form

$$\left.\begin{aligned} B_i' &= \sum_{k=1}^{f} (\beta_{ik}B_k + \sigma_{ik}\varepsilon_k) \\ \varepsilon_i' &= \sum_{k=1}^{f} (\alpha_{ik}B_k + \rho_{ik}\varepsilon_k) \end{aligned}\right\} \beta_{ik} \text{ and } \rho_{ik} \in \mathscr{D}_0;\ \alpha_{ik} \text{ and } \sigma_{ik} \in \mathscr{D}_1;\ i=1,\ldots,f. \tag{7.2}$$

These combinations with anticommuting parameters as coefficients may be defined either on the intersection $\mathscr{B} = \bigcap_{i=1}^{f} \operatorname{dom} B_i \otimes G_f$ or on their common invariant domain $C_0^\infty(\mathbb{R}^f)\otimes G_f$, where $C_0^\infty(\mathbb{R}^f)$ is a common invariant core for B_i and $B_i^\dagger$, $i=1,\ldots,f$.

On the Grassmann algebra $\mathscr{D}$ an involution is defined by

$$\begin{aligned} &\delta_i^* = \delta_i \qquad i=1,\ldots,f \\ &(\delta\delta')^* = \delta'^*\delta^* \qquad \delta^{**}=\delta \qquad \delta,\delta'\in\mathscr{D}. \end{aligned} \tag{7.3}$$

With the help of this involution, the notion of adjointness is extended to

$$B_i'^\dagger = \sum_{k=1}^{f} (\beta_{ik}^* B_k^\dagger - \sigma_{ik}^*\partial_k) \qquad \varepsilon_i'^\dagger = \sum_{k=1}^{f} (\alpha_{ik}^* B_k^\dagger + \rho_{ik}^*\partial_k) \qquad i=1,\ldots,f. \tag{7.4}$$

These definitions imply the relations

$$\begin{aligned} &[B_i', B_k'] = [B_i'^\dagger, B_k'^\dagger] = 0 \qquad \{\varepsilon_i', \varepsilon_k'\} = \{\varepsilon_i'^\dagger, \varepsilon_k'^\dagger\} = 0 \\ &[B_i', \varepsilon_k'] = [B_i'^\dagger, \varepsilon_k'^\dagger] = 0 \qquad i,k=1,\ldots,f. \end{aligned} \tag{7.5}$$

In order to conserve the CCR and CAR we use the implications

$$\begin{aligned} &[B_i', B_k'^\dagger] = \delta_{ik} I && \text{iff } \sum_{j=1}^{f} (\beta_{ij}\beta_{kj}^* + \sigma_{ij}\sigma_{kj}^*) = \delta_{ik} I \\ &\{\varepsilon_i', \varepsilon_k'^\dagger\} = \delta_{ik} I && \text{iff } \sum_{j=1}^{f} (\alpha_{ij}\alpha_{kj}^* + \rho_{ij}\rho_{kj}^*) = \delta_{ik} I \\ &[B_i', \varepsilon_k'^\dagger] = [B_i'^\dagger, \varepsilon_k'] = 0 && \text{iff } \sum_{j=1}^{f} (\beta_{ij}\alpha_{kj}^* + \sigma_{ij}\rho_{kj}^*) = 0. \end{aligned} \tag{7.6}$$

The non-zero commutators are defined on $\bigcap_{k=1}^{f} \operatorname{dom}(x_k^2 \dotplus p_k^2)\otimes G_f$ or on the common invariant core $C_0^\infty(\mathbb{R}^f)\otimes G_f$ or as forms on $\mathscr{B}$. These constraints allow us to define an isomorphism by mapping the adjoint $\varepsilon_i'^\dagger$ to the derivative from the left $\partial_i' = \partial/\partial\varepsilon_i'$ for $i=1,\ldots,f$.

The transformation (7.2) may be written in matrix notation as

$$\begin{pmatrix} B_1' \\ \vdots \\ B_f' \\ \varepsilon_1' \\ \vdots \\ \varepsilon_f' \end{pmatrix} = \begin{pmatrix} \beta_{ik} & \sigma_{ik} \\ \alpha_{ik} & \rho_{ik} \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_f \\ \varepsilon_1 \\ \vdots \\ \varepsilon_f \end{pmatrix} \qquad T = \begin{pmatrix} \beta & \sigma \\ \alpha & \rho \end{pmatrix} \tag{7.7}$$

and the constraints (7.6) can be written as a 'superunitarity' condition

$$\sum_{j=1}^{2f} T_{ij}T_{kj}^* = \delta_{ik} I \qquad T^* = \begin{pmatrix} \beta^* & \sigma^* \\ \alpha^* & \rho^* \end{pmatrix} \qquad T_{ik}^\dagger = T_{ki}^* \qquad TT^\dagger = T^\dagger T = I. \tag{7.8}$$

One may again define supercharges and a transformed Hamiltonian by

$$Q' = \sqrt{2}\sum_{i=1}^{f} \varepsilon_i' B_i' \qquad Q'^{+} = \sqrt{2}\sum_{i=1}^{f} \partial_i' B_i'^{+}$$
$$H' = \sum_{i=1}^{f} (B_i'^{+} B_i' + \varepsilon_i' \partial_i') H'^{+}. \tag{7.9}$$

Since the transformation (7.2) conserves the CCR and CAR the new operators again yield a representation of the Lie superalgebra S(2):

$$\{Q', Q'^{\dagger}\} = 2H' \qquad Q'^2 = Q'^{\dagger 2} = 0 \qquad [Q', H'] = [Q'^{\dagger}, H']0. \tag{7.10}$$

As an example, we take the case of one bosonic and one fermionic degree of freedom, $f = 1$. The constraints

$$\beta\beta^* + \sigma\sigma^* = \alpha\alpha^* + \rho\rho^* = I \qquad \beta\alpha^* + \sigma\rho^* = 0 \tag{7.11}$$

may be fulfilled by choosing

$$T = \begin{pmatrix} \beta & \sigma \\ \alpha & \rho \end{pmatrix} = \begin{pmatrix} I + \mathrm{i}\delta_1\delta_2 & \delta_1 + \mathrm{i}\delta_2 \\ -\delta_1 + \mathrm{i}\delta_2 & I - \mathrm{i}\delta_1\delta_2 \end{pmatrix} \tag{7.12}$$

then one obtains

$$Q'/\sqrt{2} = (I - 2\mathrm{i}\delta_1\delta_2)\varepsilon B - (\delta_1 - i\delta_2)B^2 \qquad Q'^{\dagger}/\sqrt{2} = (I - 2\mathrm{i}\delta_1\delta_2)\partial B^{\dagger} - (\delta_1 + \mathrm{i}\delta_2)B^{\dagger 2}$$
$$H' = H'^{\dagger} = (I + 4\mathrm{i}\delta_1\delta_2)B^{\dagger}B + (I - 4\mathrm{i}\delta_1\delta_2)\varepsilon\partial + 4\mathrm{i}\delta_1\delta_2 + 2(\delta_1 + \mathrm{i}\delta_2)\varepsilon B^{\dagger} - 2(\delta_1 - \mathrm{i}\delta_2)\partial B. \tag{7.13}$$

In this case the anticommuting parameters δ_1, δ_2 may be identified with those in the definition of supertransformations, $\delta_1 = \Theta_1$, $\delta_2 = \Theta_2$, § 6. As a matrix representation for them, one may use the Klein-Jordan-Wigner transformed Pauli matrices $\varepsilon = C_3^{\dagger}$, $\partial = C_3$, $\Theta_1 = C_1$, $\Theta_2 = C_2$. Obviously, the choice (7.12) is not the most general one.

For $f = 1$, one could also simply choose

$$T = \begin{pmatrix} I & \delta \\ -\delta & I \end{pmatrix} \qquad \delta = \delta^* \qquad \delta^2 = 0 \qquad T^{\dagger} = \begin{pmatrix} I & -\delta \\ \delta & I \end{pmatrix} \tag{7.14}$$

which implies, using $\{\varepsilon, \delta\} = \{\partial, \delta\} = 0$, that

$$B' = B + \delta\varepsilon \qquad \varepsilon' = \varepsilon - \delta B$$
$$B'^{\dagger} = B^{\dagger} - \delta\partial \qquad \partial' = \varepsilon'^{\dagger} = \partial - \delta B^{\dagger}. \tag{7.15}$$

The corresponding supercharges are

$$Q'/\sqrt{2} = \varepsilon B - \delta B^2 \qquad Q'^{\dagger}/\sqrt{2} = \partial B^{\dagger} - \delta B^{\dagger 2}$$
$$H' = B^{\dagger}B + \varepsilon\partial + 2\delta\varepsilon B^{\dagger} - 2\delta\partial B. \tag{7.16}$$

A simple choice for a superunitary transformation of f bosonic and f fermionic degrees of freedom is given by

$$B_k' = B_k + \delta_k\varepsilon_k \qquad \varepsilon_k' = \varepsilon_k - \delta_k B_k \qquad k = 1, \ldots, f. \tag{7.17}$$

Especially one may use only one anticommuting variable $\delta = \delta^*$ and take

$$T = \begin{pmatrix} I & \delta I \\ -\delta I & I \end{pmatrix} \qquad T^{\dagger} = \begin{pmatrix} I & -\delta I \\ \delta I & I \end{pmatrix}. \tag{7.18}$$

Therefore choosing $d = 1$ is always possible.

The transformation (7.17) can be performed using the supergroup of superunitary transformations

$$g(s)=\exp\left(\mathrm{i}\sum_{k=1}^{f}(s_k^* B_k^\dagger\delta_k\varepsilon_k+s_kB_k\partial_k\delta_k)\right)\qquad s_k\in\mathbb{C}$$
$$g(s)^\dagger=g(s)^{-1}=g(-s)\qquad g(s)g(s')=g(s+s') \tag{7.19}$$

on the domain $\mathscr{B}=\bigcap_{k=1}^{f}\operatorname{dom}B_k\otimes G_f$. One obtains

$$\left.\begin{aligned}g(s)B_ig(-s)&=B_i-\mathrm{i}s_i^*\delta_i\varepsilon_i\\ g(s)\varepsilon_ig(-s)&=\varepsilon_i-is_i\delta_iB_i\end{aligned}\right\}\qquad i=1,\ldots,f \tag{7.20}$$

on appropriate domains.

8. Representation of S(*N*) by self-adjoint operators

In (1.1)-(1.3) we presented an axiomatic formulation for representations of the superalgebra S(N) by N self-adjoint supercharges. Special representations can be constructed from some finite-dimensional Clifford algebras and suitable bosonic operators [21], the latter being subject to constraints in order to guarantee that the superalgebra is fulfilled.

We start with supercharges

$$Q^n=\frac{1}{\sqrt{2}}\sum_{i=1}^{f}\sum_{k=1}^{d}\gamma_{ik}A_{ik}^n\qquad A_{ik}^{n\dagger}=A_{ik}^n;\ n=1,\ldots,N;\ \gamma_{ik}=\gamma_{ik}^\dagger$$
$$\{\gamma_{ik},\gamma_{jl}\}=2\delta_{ij}\delta_{kl}I_{p'}\qquad i,j=1,\ldots,f;\ k,l=1,\ldots,d;\ p'=2^{q'};\ q'=f\cdot d/2 \tag{8.1}$$

which are self-adjoint on

$$\operatorname{dom}Q^n=\bigcap_{i=1}^{f}\bigcap_{k=1}^{d}\operatorname{dom}A_{ik}^n\otimes\mathbb{C}^{p'}\qquad n=1,\ldots,N \tag{8.2}$$

in the Hilbert space $\mathscr{H}\otimes\mathbb{C}^{p'}$. The above-mentioned constraints can be written in terms of the real Clifford algebra with negative metric

$$\{\Theta^n,\Theta^m\}=-2\delta_{nm}I_d\qquad\Theta^n=\Theta^{n*}\qquad\Theta^n\Theta^{nt}=I_d\qquad n,m=1,\ldots,N-1 \tag{8.3}$$

where the superscript t denotes the transposed matrix. The equations

$$A_{ik}^n=\sum_{j=1}^{d}\Theta_{kj}^nA_{ij}^N\qquad i=1,\ldots f;\ k=1,\ldots,d;\ n=1,\ldots,N \tag{8.4}$$

together with the 'integrability conditions'

$$\langle A_{ik}^n\psi\,|\,A_{jl}^m\phi\rangle-\langle A_{jl}^m\psi\,|\,A_{ik}^n\phi\rangle=\langle A_{jl}^n\psi\,|\,A_{ik}^m\phi\rangle-\langle A_{ik}^m\psi\,|\,A_{jl}^n\phi\rangle \tag{8.5}$$

for all $\phi,\psi\in\bigcap_{i=1}^{f}\bigcap_{k=1}^{d}\operatorname{dom}A_{ik}^n=\mathscr{A}$ and $i,j=1,\ldots,f,\ k,l=1,\ldots,d,\ m,n=1,\ldots,N$ and $m\neq n$ imply the superalgebra S(N) in the sense of forms. Here we have used the fact that each linear combination in (8.4) contains just one term, which implies that $\operatorname{dom}Q^1=\ldots=\operatorname{dom}Q^N$, as postulated in (1.1).

The irreducible representations of the Clifford algebras (8.3) are all explicitly tabulated in the literature [21], they occur modulo 8, and one finds the ten possibilities which are listed in table 1.

Table 1. The ten possibilities of the irreducible representations of the Clifford algebras (8.3).

N	d	r	p
2	2	1	2
3	4	1	4
4	4	2	4
5	8	1	16
6	8	1	16
7	8	1	16
8	8	2	16
9	16	1	16^2

Here r denotes the number of inequivalent irreducible representations, $p = 2^q$, $q = d/2$.

The f degrees of freedom, which were introduced in (8.1), are obtained as an f-fold direct sum of such Clifford algebras of type (8.3), each of them occurs with the same d, but dependent on N.

8.1. Case N=2

With the definitions

$$\begin{aligned} &\varepsilon_k = (\gamma_{k1} + \mathrm{i}\gamma_{k2})/2 && \partial_k = (\gamma_{k1} - \mathrm{i}\gamma_{k2})/2 = \varepsilon_k^\dagger \\ &G_k = (A_{k1}^1 - \mathrm{i}A_{k2}^1)/\sqrt{2} && k = 1, \ldots, f \end{aligned} \tag{8.6}$$

where G_k are assumed to be closed (see necessary and sufficient conditions (4.4)), one obtains the model which was discussed in § 4. Assumption (8.2) is fulfilled for this model if dom G_k = dom $G_k^\dagger$ for $k = 1, \ldots, f$; then

$$G_k^\dagger = (A_{k1}^1 + \mathrm{i}A_{k2}^1)/\sqrt{2} \qquad \text{for } k = 1, \ldots, f. \tag{8.7}$$

The Clifford algebra (8.3) contains one element, say $\Theta^1 = \mathrm{i}\sigma^2$. The special choice $A_{k1}^1 = V_k$, $A_{k1}^2 = \pi_k$, $k = 1, \ldots, f$, was investigated in § 4. The integrability conditions (8.5) imply that the assumption (4.21) is fulfilled.

8.2. Case N=4

We use the irreducible representation

$$\Theta^1 = \sigma^3 \otimes \mathrm{i}\sigma^2 \qquad \Theta^2 = \mathrm{i}\sigma^2 \otimes I_2 \qquad \Theta^3 = \sigma^1 \otimes \mathrm{i}\sigma^2 \tag{8.8}$$

an inequivalent representation is obtained by permuting Θ^1 and Θ^2. A generalisation of the model, which was presented in § 4, is obtained by assuming that the following sums of operators are closed:

$$\begin{aligned} &G_{ik}^n = (A_{ik}^n - \mathrm{i}A_{i,k+2}^n)/\sqrt{2} && n, k = 1, 2;\ i = 1, \ldots, f \\ &\varepsilon_{ik} = (\gamma_{ik} + \mathrm{i}\gamma_{i,k+2})/2 && \partial_{ik} = (\gamma_{ik} - \mathrm{i}\gamma_{i,k+2})/2 = \varepsilon_{ik}^\dagger. \end{aligned} \tag{8.9}$$

Denoting $Q^n = Q_n^1$ and $Q^{n+2} = Q_n^2$ for $n = 1, 2$, we get the four self-adjoint supercharges

$$\begin{aligned} &Q_n^1 = \sum_{i=1}^{f} \sum_{k=1}^{2} (\varepsilon_{ik} G_{ik}^n + \partial_{ik} G_{ik}^{n\dagger}) \\ &Q_n^2 = \mathrm{i} \sum_{i=1}^{f} \sum_{k=1}^{2} (\partial_{ik} G_{ik}^{n\dagger} - \varepsilon_{ik} G_{ik}^n). \end{aligned} \qquad n = 1, 2 \tag{8.10}$$

Here again we need that dom $G^n_{ik} = \text{dom } G^{n\dagger}_{ik}$ in order to fulfil (8.2). This implies that

$$iG^{1\dagger}_{i1} = G^2_{i2} \qquad G^{1\dagger}_{i2} = iG^2_{i1} \qquad i = 1, \ldots, f. \tag{8.11}$$

In order to investigate whether the supercharges Q^n, $n = 1, \ldots, N$, are self-adjoint on the domain (8.2) for some special models [21], one may again use the KLMN theorem [32, 33] and apply theorems of Wüst [33, 34] or of Kato-Rellich [32-34] to treat perturbations.

If the estimate

$$\left| \sum_{\substack{i,i'=1 \\ i \neq i' \text{ and/or } k \neq k'}}^{f} \sum_{k,k'=1}^{d} \langle \gamma_{ik} A^n_{ik} \Psi \mid \gamma_{i'k'} A^n_{i'k'} \Psi \rangle \right| \leqslant \alpha \sum_{i=1}^{f} \sum_{k=1}^{d} \langle A^n_{ik} \Psi \mid A^n_{ik} \Psi \rangle \tag{8.12}$$

holds for all $\Psi \in \mathscr{A} \otimes \mathbb{C}^{p'}$ with $0 < \alpha < 1$, Q^n is closed. If the non-negative operator

$$H_0 = \tfrac{1}{2} \sum_{i=1}^{f} \sum_{k=1}^{d} (A^n_{ik})^2 = H^\dagger_0 \geqslant 0 \tag{8.13}$$

defined as form sum on dom $H_0^{1/2} = \mathscr{A}$ is essentially self-adjoint on some suitable core C_0, and if $(Q^n)^2 - H_0$ is relatively bounded with respect to H_0 with relative bound $\gamma \leqslant 1$, then $(Q^n)^2$ is essentially self-adjoint on C_0. In the case of self-adjoint supercharges Q^n, $n = 1, \ldots, N$, for $\gamma < 1$, the corresponding Hamilton operator is given by

$$H = (Q^n)^2 = H_0 + \left(\sum_{\substack{i,i'=1 \\ i<i'}}^{f} \sum_{k,k'=1}^{d} + \sum_{\substack{i,i'=1 \\ i=i'}}^{f} \sum_{\substack{k,k'=1 \\ k<k'}}^{d} \right) \gamma_{ik} \gamma_{i'k'} [A^n_{ik}, A^n_{i'k'}] \tag{8.14}$$

$$\text{dom } H = \text{dom } H_0 \qquad \text{dom } H^{1/2} = \text{dom } Q^n = \text{dom } H_0^{1/2} \qquad n = 1, \ldots, N.$$

8.3. *Cases $N = 4$, 6 and 8*

Here, supersymmetry transformations may be defined as direct products of transformations of the type (6.4). With the definitions

$$\begin{aligned} &Q^1_n = Q^n = Q^{n\dagger} \qquad Q^2_n = Q^{n+N/2} = Q^{n+N/2\dagger} \\ &Q_n = (Q^1_n + iQ^2_n)/\sqrt{2} \qquad Q^\dagger_n \supseteq (Q^1_n - iQ^2_n)/\sqrt{2} \qquad Q^{\dagger\dagger}_n = \bar{Q}_n \end{aligned} \tag{8.15}$$

on

$$\text{dom } Q^\dagger_n \supseteq \text{dom } Q_n = \text{dom } Q^1_n = \text{dom } Q^2_n = \text{dom } H^{1/2} \qquad n = 1, \ldots, N/2$$

the superalgebra S(N) may be rewritten as

$$\begin{aligned} &\langle Q^\dagger_n \Psi \mid Q^\dagger_m \Phi \rangle + \langle Q_m \Psi \mid Q_n \Phi \rangle = 2\delta_{nm} \langle H^{1/2} \Psi \mid H^{1/2} \Phi \rangle \\ &\langle Q^\dagger_n \Psi \mid Q_m \Phi \rangle + \langle Q^\dagger_m \Psi \mid Q_n \Phi \rangle = 0 \qquad \Psi, \Phi \in \text{dom } H^{1/2} \\ &\langle Q^\dagger_n \Psi' \mid H \Phi' \rangle = \langle H \Psi' \mid Q_n \Phi' \rangle \qquad \Psi', \Phi' \in \text{dom } H \qquad n, m = 1, \ldots, N/2. \end{aligned} \tag{8.16}$$

These anticommutation relations can be converted to commutation relations analogously as for the case $N=2$ (6.3) using anticommuting parameters $\Theta^n = \Theta_1^n + i\Theta_2^n$, $\Theta^{n*} = \Theta_1^n - i\Theta_2^n$, $\Theta_k^{n*} = \Theta_k^n$, $k=1,2$, $n=1,\ldots,N/2$:

$$\langle \theta^{n*} Q_n^\dagger \Psi \mid \Theta^{m*} Q_m^\dagger \Phi\rangle - \langle Q_m \Theta^m \Psi \mid Q_n \Theta^n \Phi\rangle = 2\delta_{nm}\Theta^n\Theta^{m*}\langle H^{1/2}\Psi \mid H^{1/2}\Phi\rangle$$

$$\langle \Theta^{n*} Q_n^\dagger \Psi \mid Q_m \Theta^m \Phi\rangle = \langle \Theta^{m*} Q_m^\dagger \Psi \mid Q_n \Theta^n \Phi\rangle \qquad \text{for } \Phi, \Psi \in \operatorname{dom} H^{1/2} \tag{8.17}$$

$n, m = 1, \ldots, N/2$.

The corresponding group of supertransformations defined in the sense of forms on dom $H^{1/2}$ by

$$\begin{aligned} g_N(t, s_1, \ldots, s_{N/2}, r_1, \ldots, r_{N/2}) &= \exp\left(itH + i \sum_{n=1}^{N/2} (s_n Q_n \Theta^n + s_n^* \Theta^{n*} Q_n^\dagger + r_n H \Theta^n \Theta^{n*})\right) \\ &= \exp(itH)\, g(0, s_1, r_1) \ldots g(0, s_{N/2}, r_{N/2}) \qquad t, r_n \in \mathbb{R} \qquad s_n \in \mathbb{C} \end{aligned} \tag{8.18}$$

turns out to be the direct product of $N/2$ supergroups of the type (6.4), and superunitarity holds in the sense that

$$\begin{aligned} g_N(t, s_1, \ldots, s_{N/2}, r_1, \ldots, r_{N/2})^\dagger &= g_N(t, s_1, \ldots, s_{N/2}, r_1, \ldots, r_{N/2})^{-1} \\ &= g_N(-t, -s_1, \ldots, -s_{N/2}, -r_1, \ldots, -r_{N/2}). \end{aligned} \tag{8.19}$$

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